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Stability of plane Poiseuille flow of a dilute suspension of slender fibres

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The linear hydrodynamic stability problem for plane Poiseuille flow of a dilute suspension of rigid fibres is solved numerically. The constitutive equation given by Batchelor (1970*a*, *b*, 1971) is used to model the rheological properties of the suspension. The resulting eigenvalue problem is shown to be singular. The appropriate contour in the complex plane is determined by considering an initial-value problem. It is shown that, for a fixed, but not too large, inclination of the wave front to the mean flow, the fibres cause the critical Reynolds number to increase monotonically with the product of the volume fraction of the fibres and the square of their aspect ratio. The stabilizing influence of the fibres seems to vanish for large wave inclination angles.

1. Introduction

There are several reasons for studying the fluid mechanics of fibre suspensions. Flows of such suspensions are common in industrial applications, e.g. in chemical and pulp engineering. Furthermore, it is known that addition of small amounts of certain fibres to a fluid may reduce its drag in turbulent flows (Hoyt 1972*a*; Vaseleski & Metzner 1974). As is well known, a similar effect can be obtained by addition of certain long-chain polymer molecules (Virk 1975). It has been speculated whether the polymer molecules in a drag-reducing fluid are uncoiled into long threads by the turbulent straining field, whereby their hydrodynamic effect would be similar to that of suspended fibres (Landahl 1972; Lumley 1972; Landahl & Bark 1974; Batchelor 1976). Knowledge of the hydrodynamic properties of fibre suspensions may thus, in some cases, also be applicable to the rheologically far more complex polymer solutions.

In this paper the linear hydrodynamic stability problem for a plane Poiseuille flow of a dilute suspension of rigid fibres is solved. The constitutive equation derived by Batchelor (1970*a*, *b*, 1971) is used to model the rheological properties of the suspension. It should be pointed out that the present work is not an attempt to explain turbulent drag reduction due to fibre additives, but is merely intended to shed some light on the general hydrodynamic stability properties of fibre suspensions as such. To study the effect of fibre additives on turbulent flows, one should instead investigate the Kelvin–Helmholtz type of instability, which presumably is a crucial mechanism in the turbulent bursting process (Kim, Kline & Reynolds 1971). A two-dimensional stability problem of this kind was studied by Landahl & Bark (1974), who found that such instabilities are suppressed by fibre additives, if the rheology of the suspension is modelled according to Batchelor (1970*a*, *b*, 1971). A visual study by Filipsson, Lagerstedt & Bark (1977) of jets in a liquid at rest, which is a flow susceptible to Kelvin–Helmholtz instability, showed the same trend as the theoretical results given by Landahl & Bark (1974) if fibres were added to the fluid in the jet. The same effect for polymers was found by Gadd (1965). This indicates that the hydrodynamic effects of the fibres in this flow were similar to those of the polymers.

As far as transition from laminar to turbulent flow is concerned, results from the kind of calculations presented in the present work should be interpreted with care. It is known that the final stage of transition involves rapidly growing small-scale motions superposed on a rather slowly growing and nonlinearly distorted Tollmien–Schlichting wave. This has been observed both for the Blasius boundary layer (Klebanoff, Tidstrom & Sargent 1962) and for plane Poiseuille flow (Nishioka, Iida & Ichikawa 1975). However, the presence of a slowly growing Tollmien–Schlichting wave of the kind calculated in this work seems to be a prerequisite for transition to take place.

There are no experimental data available on transition in a plane Poiseuille flow of a fibre suspension. For Poiseuille flow in a pipe, it was shown by Vaseleski & Metzner (1974) that the Reynolds number for transition for a suspension containing very slender asbestos fibres may be significantly larger than that for the Newtonian solvent. Although the processes causing transition in plane and axisymmetric Poiseuille flow are presumably qualitatively similar, several points regarding this matter remain to be resolved (Sarpkaya 1975). It should also be pointed out that some experiments have shown that transition occurs at a higher Reynolds number in pipe flow of certain polymer solutions if the polymer molecules have a comparatively large molecular weight, and therefore a comparatively large relaxation time (Hoyt 1972*b*). A large relaxation time will decrease the resistance of the molecule to being straightened out by the flow. One may thus speculate whether polymer molecules have the same effect as fibres in these cases also. In most cases, however, polymer additives do not affect the transition Reynolds number.

Using another rheological model for a fibre suspension, Gyr (1977) solved the same problem as is treated in the present work. It seems to us that the rheological model used by Gyr (see Henkel & Gyr 1977) is somewhat oversimplified. However, Gyr's results show the same trend as ours.

The mathematical formulation of the stability problem is given in § 2. It is shown that the resulting equation is singular at the critical layer, i.e. at the point in the flow where the phase speed of a neutrally stable perturbation wave is the same as the mean velocity. The structure of this singularity and its mathematical consequences are discussed in § 3. In § 4, the numerical method used is briefly described. The stability boundaries computed are presented in § 5.

2. Statement of the problem

The problem to be considered is the linear hydrodynamic stability of a steady parallel flow of a dilute suspension of rigid fibres between two infinitely large parallel walls. To model the rheological properties of the suspension, the constitutive equation given by Batchelor (1970*a*, *b*, 1971) will be used. In this model, the direction of the suspended fibres is described by a vector field $\mathbf{p}(\mathbf{x}, t)$, where $\mathbf{x}$ is the position vector and

t is the time. The stress tensor, in Cartesian co-ordinates, is given in terms of the vector field $\mathbf{p}(\mathbf{x}, t)$ and the velocity field $\mathbf{v}(\mathbf{x}, t)$ by

$$t_{kl} = 2\mu d_{kl} + \frac{\mu \Phi r^2}{\ln 2r - \frac{3}{2}} p_k p_l p_m p_n d_{mn}, \tag{2.1}$$

where

$$d_{kl} = \tfrac{1}{2}(v_{k,l} + v_{l,k}) \tag{2.2}$$

is the rate-of-deformation tensor, μ the dynamic shear viscosity of the Newtonian solvent, Φ the volume fraction of the suspended particles and r their aspect ratio. The first term in (2.1) is the contribution to the stress from the solvent and the second term gives the contribution from the suspended fibres. In asbestos-fibre suspensions, the aspect ratio r may be very large. Estimated values as large as 10^4 have been reported for certain fibres (Rosinger, Woodhams & Chaffey 1974). This means that the non-dimensional parameter

$$B = \Phi r^2/(\ln 2r - \tfrac{3}{2}), \tag{2.3}$$

which enters the part of the stress tensor due to the fibres [see (2.1)], may well be of order unity or larger even if the volume fraction Φ is small. This means that, in some flows, the stress caused by the fibres can dominate the usual viscous stress in the solvent even in dilute suspensions. There is experimental evidence for this in nearly irrotational flows (Mewis & Metzner 1974; Kizior & Seyer 1974). It should be pointed out, though, that the assumption of rigid fibres may not be a very good one for asbestos fibres. There is also a very large spread of fibre lengths in commercial asbestos-fibre samples. The assumption of small fibres, which is implicitly made when using (2.1), may therefore be questionable in some cases.

The vector field $\mathbf{p}$ can be calculated in terms of the velocity field $\mathbf{v}$ from the following set of equations:

$$p_{k,t} + v_m p_{k,m} = \omega_{km} p_m + \frac{r^2-1}{r^2+1}(d_{km} p_m - d_{mn} p_m p_n p_k), \tag{2.4a}$$

$$p_k p_k = 1, \tag{2.4b}$$

where

$$\omega_{kl} = \tfrac{1}{2}(v_{k,l} - v_{l,k}) \tag{2.5}$$

is the rotation-rate tensor. Equation (2.4b) ensures that the lengths of the fibres remain constant. Having specified the constitutive equation for the suspension, its motion can be calculated from the momentum equation

$$\rho(v_{k,t} + v_m v_{k,m}) = -\pi_{,k} + t_{lk,l}, \tag{2.6}$$

where ρ is the density of the solvent and π the pressure, and the continuity equation

$$v_{k,k} = 0. \tag{2.7}$$

The calculated velocity field must satisfy the no-slip condition at the plates, i.e.

$$v_k = 0 \quad \text{for} \quad x_2 = 0, 2L, \tag{2.8}$$

where it has been assumed that the x_2 direction is perpendicular to the plates and that the plates are at $x_2 = 0$ and $x_2 = 2L$, respectively. The flow, whose linear stability is to be studied, is assumed to be driven by a constant pressure gradient in the x_1 direction. If one assumes that the resulting flow is a parallel shear flow, one can show from (2.4a) that

$$p_k = \delta_{k1}, \tag{2.9}$$

which means that the fibres are all aligned in the direction of the mean flow. It can also be shown that, if their direction is given by (2.9), the suspended fibres have no dynamical effect at all. The resulting flow is thus a standard Poiseuille flow of the form

$$v_k = U_0[2x_2/L - (x_2/L)^2]\,\delta_{k1}, \tag{2.10}$$

where U_0 is the centre-line velocity. For notational convenience, in what follows v_k will be written as

$$v_k = U(x_2)\,\delta_{k1}. \tag{2.11}$$

From here on it is assumed that all quantities are non-dimensionalized by L (lengths), U_0 (velocities), L/U_0 (times) and ρU_0^2 (pressures and stresses per unit area).

In order to study the linear stability of the flow given by (2.10), the dependent variables are decomposed into a part of order unity and a small perturbation according to

$$v_k = U(x_2)\,\delta_{k1} + v_k'(x_r, t), \tag{2.12a}$$

$$\pi = -2x_1/R + \pi'(x_r, t), \tag{2.12b}$$

$$p_k = \delta_{k1} + p_k'(x_r, t), \tag{2.12c}$$

$$t_{km} = (2/R)\,(1-x_2)\,(\delta_{k1}\,\delta_{m2} + \delta_{k2}\,\delta_{m1}) + t_{km}'(x_r, t), \tag{2.12d}$$

where

$$R = \rho U_0 L/\mu \tag{2.13}$$

is the Reynolds number and the perturbation quantities are denoted by primes. These primes are dropped in what follows, since no confusion will arise thereby.

It can be shown from (2.4a) that, for the type of mean flow given by (2.11), to first order in the perturbation amplitude the only non-zero perturbation component of the vector field $\mathbf{p}$ is p_2, which is to be calculated from

$$p_{2,t} + Up_{2,1} = v_{2,1}. \tag{2.14}$$

It then follows from (2.14) and (2.1) that the only perturbation stress caused by the suspended fibres is a normal stress in the x_1 direction. The perturbation stress tensor can, after some algebra, be calculated as

$$t_{kl} = R^{-1}(v_{k,l} + v_{l,k}) + BR^{-1}(p_2 DU + v_{1,1})\,\delta_{k1}\,\delta_{l1}. \tag{2.15}$$

For simplicity, the following changes in notation are made:

$$(v_1, v_2, v_3) \rightarrow (u, v, w), \quad (x_1, x_2, x_3) \rightarrow (x, y, z). \tag{2.16a, b}$$

As in the classical theory of linear hydrodynamic stability (Lin 1955), the eventual exponential growth in time of a wave disturbance will be studied. The perturbation quantities are therefore assumed to be of the form

$$(u, v, w, \pi, p_2) = (\hat{u}, \hat{v}, \hat{w}, \hat{\pi}, \hat{p}_2)\exp\{i(\alpha x + \beta z - \alpha c t)\}, \tag{2.17}$$

where variables with a caret depend on y only. α and β are the wavenumbers in the x and z directions, respectively, and c is the phase speed in the x direction. In this work it will be assumed that α and β are real. The phase speed c will, in general, be complex and the sign of its imaginary part c_I will determine the stability of the flow.

Substituting (2.17*a*) into (2.14), (2.15) and (2.6), (2.7) gives the following set of equations:

$$i\alpha(U-c)\hat{p}_2 = i\alpha\hat{v}, \tag{2.18a}$$

$$i\alpha(U-c)\hat{u}+(DU)\hat{v} = -i\alpha\hat{\pi}+R^{-1}(D^2-k^2)\hat{u}+BR^{-1}(\hat{p}_2DU+i\alpha\hat{u}), \tag{2.18b}$$

$$i\alpha(U-c)\hat{v} = -D\hat{\pi}+R^{-1}(D^2-k^2)\hat{v}, \tag{2.18c}$$

$$i\alpha(U-c)\hat{w} = -i\beta\hat{\pi}+R^{-1}(D^2-k^2)\hat{w}, \tag{2.18d}$$

$$i\alpha\hat{u}+D\hat{v}+i\beta\hat{w} = 0, \tag{2.18e}$$

where

$$D = d/dy, \quad k^2 = \alpha^2+\beta^2. \tag{2.19a, b}$$

Equations (2.18*a*–*e*) are to be solved subject to

$$\hat{u} = \hat{v} = \hat{w} = 0 \quad \text{for} \quad y = 0, 2. \tag{2.20}$$

If α, β, B and R are given, (2.18*a*–*e*) and (2.20) define an eigenvalue problem for c. If $\hat{v}$ is non-zero at the point in the complex y plane where $U = c$, which will be called y_c, the system of equations (2.18*a*–*e*) will have a singular point at $y = y_c$, as can be seen from (2.18*a*, *b*). It will be shown below that $\hat{v}(y_c)$ is non-zero. This singularity means that, for a real y_c, the tilting of the fibres will be infinite at $y = y_c$. This results from a discrepancy in the constitutive equation used, which in principle can be removed by including some diffusive mechanism in (2.4*a*), which describes the motion of the fibres in a given velocity field. However, there is at present no physically realistic way of incorporating such effects (E. J. Hinch, private communication). Note that singularities of this kind also appear in the linear stability problem for an inviscid fluid (Lin 1955, chap. 8). The mathematical consequences of the singularity in the present case will be discussed in somewhat more detail in the next section.

By introducing the transformation (Squire 1933)

$$\tilde{u} = (\alpha\hat{u}+\beta\hat{w})/k, \quad \tilde{w} = (\beta\hat{u}-\alpha\hat{w})/k, \tag{2.21a, b}$$

one can, after some algebra, derive from (2.18*a*–*e*) the following set of equations for $\tilde{u}$, $\hat{v}$ and $\tilde{w}$:

$$(U-c)(D^2-k^2)\hat{v}-(D^2U)\hat{v}+\frac{i}{\alpha R}(D^2-k^2)^2\hat{v}+\frac{i\alpha B}{R}D\left[\frac{(DU)\hat{v}}{U-c}-\frac{\alpha^2}{k^2}D\hat{v}+\frac{i\alpha\beta}{k}\tilde{w}\right] = 0, \tag{2.22a}$$

$$(U-c)\tilde{w}+\frac{i}{\alpha R}(D^2-k^2)\tilde{w}-\frac{i\beta(DU)}{\alpha k}\hat{v}-\frac{\beta B}{kR}\left[\frac{(DU)\hat{v}}{U-c}-\frac{\alpha^2}{k^2}D\hat{v}+\frac{i\alpha\beta}{k}\tilde{w}\right] = 0, \tag{2.22b}$$

$$ik\tilde{u}+D\hat{v} = 0, \tag{2.22c}$$

which are to be solved subject to

$$\hat{v} = 0, \quad \tilde{u} = (i/k)D\hat{v} = 0 \quad \text{for} \quad y = 0, 2, \tag{2.23a, b}$$

$$\tilde{w} = -\frac{ik}{\alpha^3\beta B(U-e)}\left[\frac{i}{\alpha R}D^5\hat{v}+\left(U-c-\frac{i2k^2}{\alpha R}-\frac{i\alpha^3B}{k^2R}\right)D^3\hat{v}\right.$$
$$\left.+DU\left(1+\frac{i\alpha B}{R(U-c)}\right)D^2\hat{v}\right] = 0 \quad \text{for} \quad y = 0, 2, \tag{2.23c}$$

where

$$e = c+\frac{i}{R}\left(\frac{\alpha\beta^2B}{k^2}+\frac{k^2}{\alpha}\right). \tag{2.23d}$$

Note that (2.23*c*) has no meaning for $B = 0$. The possibility that the factor $(U-e)^{-1}$ in (2.23*c*) is infinite will be discussed in the next section. It should also be noted that $\tilde{w}$ appears in (2.22*a*). This is not the case in the corresponding stability problem for a Newtonian fluid ($B = 0$), which leads to an ordinary differential equation of fourth order in y. In the present case, it appears that there is no linear transformation of the velocity components which gives a differential equation of order lower than six. For a fibre suspension modelled according to (2.1) and (2.4*a*, *b*), there is thus no Squire theorem (Squire 1933) which reduces a three-dimensional stability problem to a two-dimensional one.

3. Structure of the mathematical problem

With some labour, one can derive from (2.22*a*, *b*) a sixth-order equation for $\hat{v}$:

$$\sum_{n=0}^{6} a_n D^{(6-n)}\hat{v} = 0, \tag{3.1}$$

where the coefficients a_n are of the form

$$a_0 = 1, \quad a_1 = g_1(y)/(U-e), \quad a_2 = g_2(y), \tag{3.2a–c}$$

$$a_k = g_k(y)/(U-e)\,(U-c)^{k-2}, \quad k = 3, \ldots, 6. \tag{3.2d}$$

The functions g_k appearing in (3.2*b*–*d*) are analytic functions of y. These functions are rather complicated and are given in the appendix. The algebra was checked by deriving (3.1) both by hand and by using the REDUCE system (Hearn 1971) for automatic symbolic manipulation on a computer. There are obviously two singular points in (3.1). These occur at values of y such that $U = e$, where e is defined by (2.23*d*), and $U = c$ respectively. In the neighbourhood of the point $y = y_e$, where y_e is defined by $U(y_e) = e$, one can show that (3.1) has six regular, linearly independent solutions.† The proof of this is somewhat lengthy and is therefore not reproduced here.

Near $y = y_c$, a straightforward Frobenius expansion in terms of the variable

$$\xi \equiv y - y_c \tag{3.3}$$

gives the following behaviour of the six linearly independent solutions:

$$\hat{v}_l = \xi^{(6-l)} + \frac{i\nu}{7-l}\xi^{(7-l)} + O(\xi^{(8-l)}), \quad l = 1, \ldots, 5, \tag{3.4a}$$

$$\hat{v}_6 = 1 + i\nu\xi - \left[\frac{\alpha^2 B}{2}\ln\xi + \tfrac{1}{4}\left(10\nu^2 + \frac{i\nu D^2U_c}{DU_c} - 3k^2 + \frac{61\alpha^2 B}{9}\right)\right]\xi^2 + O(\xi^3\ln\zeta), \tag{3.4b}$$

where

$$\nu = \alpha k^2 R DU_c/(k^4 + \alpha^2\beta^2 B), \tag{3.4c}$$

$$D^n U_c = D^n U(y)_{y=y_c}, \quad n = 1, 2. \tag{3.4d}$$

Logarithmic terms will also appear to higher order in (3.4*a*). Because of the presence of logarithmic terms in (3.4*a*, *b*), one must decide how to choose the proper path of integration in the complex y plane. The same kind of problem was resolved by Lin (1955, chap. 8) in his treatment of the linear hydrodynamic stability problem for an

† The authors are obliged to Professor L. N. Howard for the proof of this.

inviscid fluid. Lin determined the proper path by comparing the singular inviscid solution with the asymptotic limit, for vanishing viscosity, of the solution to the corresponding viscous problem. As was discussed in the previous section, this procedure can not be carried out in the present problem. It should be pointed out, however, that we are implicitly assuming that in reality there are mechanisms, presumably of a diffusive nature, preventing the unphysical behaviour of the suspension which emerges as a mathematical result when the constitutive equation given by (2.1) and (2.4a, b) is used for the present problem. Such mechanisms, which should be weak everywhere except near y_c, are, unfortunately, absent from (2.4a).

One may alternatively determine the correct path of integration by considering an initial-value problem in which all disturbances are zero before a specified time. This was done for an inviscid fluid by Dikii (1960). In that case, it was shown that the proper path in the complex y plane passes below the point $y = y_c$ if

$$DU_c > 0 \quad \text{for} \quad y_c \quad \text{real} \tag{3.5a}$$

and above if

$$DU_c < 0 \quad \text{for} \quad y_c \quad \text{real.} \tag{3.5b}$$

Inequalities (3.5a, b) were, of course, also obtained by Lin (1955, chap. 8). If neither (3.5a) nor (3.5b) is satisfied, the initial-value problem cannot be posed. Because the mathematical structure of the solution near $y = y_c$ in the problem dealt with by Dikii (1960) is essentially of the same kind as in the present problem, it is easily shown that the rules (3.5a, b) must be used for the present problem as well. These rules can be trivially extended to complex y_c. The proper path of integration is thus determined.

Owing to the algebraic complexity of (3.1), no analytic solutions could be found and numerical integration had to be used. Before presenting some results, a few details about the numerical procedure used will be given.

4. Numerical procedure

In order to calculate the stability boundary, i.e. the wavenumbers α and β for which $c_I = 0$, for given values of B and R, one may in principle integrate (3.1) along the real y axis for amplified waves and extrapolate the results to $c_I = 0$. This has been done in two-dimensional calculations of Kelvin–Helmholtz instability of a fibre suspension (Landahl & Bark 1974). For this kind of instability the real and imaginary parts of the phase speed, c_R and c_I respectively, are of the same order of magnitude, which, in practice, turns out to be necessary for extrapolation to $c_I = 0$ to give accurate results. However, for Poiseuille flow c_I is much smaller than c_R and extrapolation does not work. In order to be able to calculate neutrally stable waves, the path of integration was deformed into the lower half of the complex y plane. The contour used was a circular arc, crossing the real axis at $y = 0$ and $y = 1$.

The numerical integration scheme used was a fourth-order Runge–Kutta scheme. For $\beta = 0$, one can show that two of the solutions of (2.22a, b) have a moderate variation and that the other two grow rapidly as y approaches either of the walls. For $\beta \neq 0$, there are two moderately varying solutions and four rapidly growing ones. To suppress merging of the rapidly growing solutions with the slowly growing ones during the numerical integration, the method of orthonormalization (Godunov 1961) was used.

Owing to the rather complex algebraic structure of (3.1), some extensive testing of the computer program was carried out in order to eliminate algebraic errors as much as possible. The program was tested in the following ways.

(i) The numerical solution was compared with the exact one for $U = 1$.

(ii) A separate program was written in which (2.18*a*–*e*) were integrated as a system. The results were then compared with those obtained by integration of (3.1).

(iii) Another separate program was written, in which the adjoint problem to (3.1) was solved. The eigenvalues of the adjoint problem were then compared with those obtained from (3.1).

In these tests three-digit agreement was obtained in all cases. The eigenvalues for $\beta = 0$, which are to be computed from a fourth-order equation [cf. (2.22*a*)], were computed by using a separate program.

Some extensive numerical experiments were carried out using different step lengths and different circular arcs with the same end points. The step length was halved until no change in the first four digits of the eigenvalue could be detected. The first four digits of the eigenvalues were found to be the same for different arcs.

5. Results and discussion

Some neutral-stability curves are shown in figures 1 and 2 for the most unstable wave mode. This mode is symmetric with respect to the centre of the channel. In each of these figures, the angle θ between the wave front and the direction of the mean flow, defined by

$$\theta = \arctan(\beta/\alpha), \tag{5.1}$$

is kept constant. The diagrams show the streamwise wavenumber α of the neutrally stable wave as a function of R for different values of B. Note that the R axis is displaced to the left for the highest values of θ.

For $\theta = \frac{1}{6}\pi$, the fibres are stabilizing in the sense that the critical Reynolds number is increased. The stabilizing effect, for a fixed θ, is monotone in B. For the non-zero values of B shown in figures 1 (*a*) and (*b*), however, the critical Reynolds number is decreased for the higher value of θ, while the opposite is true for the Newtonian case $B = 0$. As can be seen from figures 1 (*c*) and 2, the effect of the fibres disappears for $\theta \gtrsim \frac{1}{3}\pi$. This result is also illustrated in table 1, where some points of neutral stability in the α, R plane are shown for various values of B and $\theta = \frac{17}{36}\pi$. Results for other velocity profiles, to be published elsewhere, showed that the merging of the neutral-stability curves shown in figure 2 does not occur in general and is presumably due to some specific property of the mean velocity profile investigated in this work.

Figures 3 (*a*) and (*b*) show the effect of the fibres on the complex phase speed as function of α for different fixed B, θ and R. According to figure 3 (*a*), the fibres affect the real part of the phase speed very weakly. This was found to be the case for all eigenvalues computed in this investigation. Figure 3 (*b*) shows the monotone decrease in c_I for increasing B and fixed values of θ and R.

For the sake of completeness, it may be of interest to have some information about the effect of the fibres on the higher wave modes, which for the Newtonian case are more stable than the one discussed above. Figure 4 shows c_I as function of B, for fixed α, β and R, for the second least stable mode. This mode is antisymmetric with respect to the centre of the channel. As can be seen from figure 4, there is a feeble

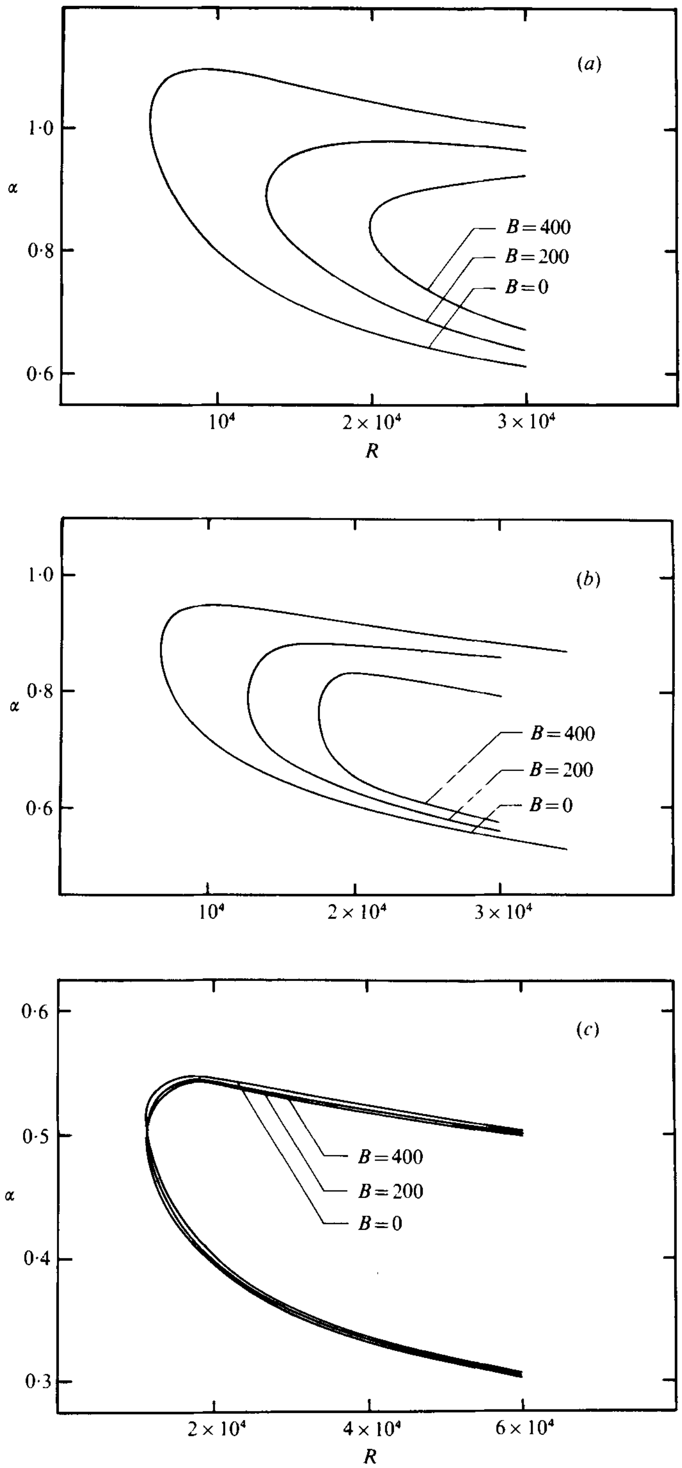

FIGURE 1. Neutral-stability curves for (*a*) $\theta = 0$, (*b*) $\theta = \frac{1}{6}\pi$ and (*c*) $\theta = \frac{1}{3}\pi$.

B	R	α
0	73500	0·0818
		0·0936
	115000	0·0695
		0·0953
200	73500	0·0817
		0·0937
	115000	0·0695
		0·0953
400	73500	0·0815
		0·0937
	115000	0·0695
		0·0953

TABLE 1. Some points of neutral stability in the α, R plane for $\theta = \frac{17}{36}\pi$.

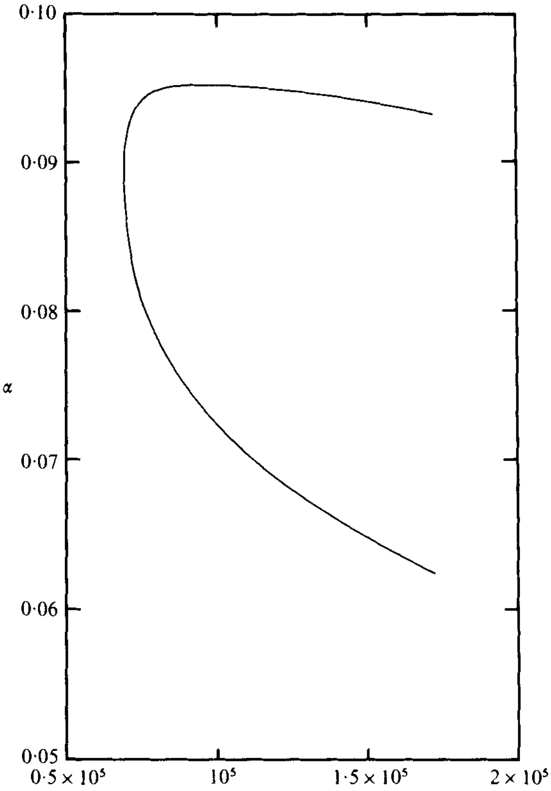

FIGURE 2. Neutral-stability curves for $\theta = \frac{17}{36}\pi$, $B = 0, 200, 400$.

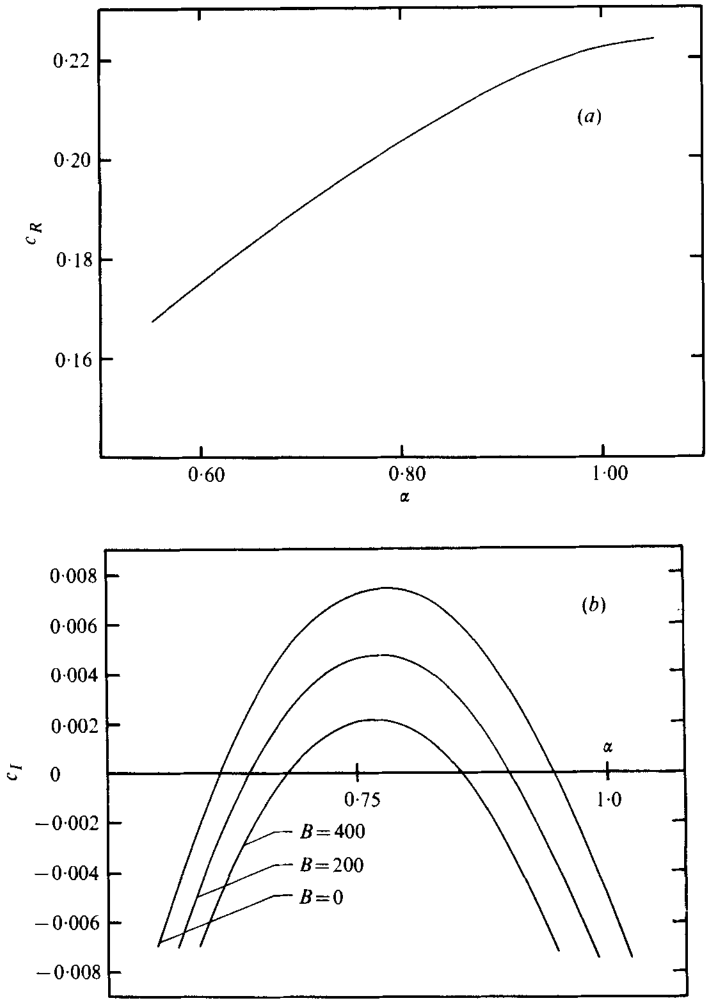

FIGURE 3. (*a*) Phase velocity c_R and (*b*) c_I as function of α for $\beta = 0{\cdot}5\,\alpha$ and $R = 2 \times 10^5$.

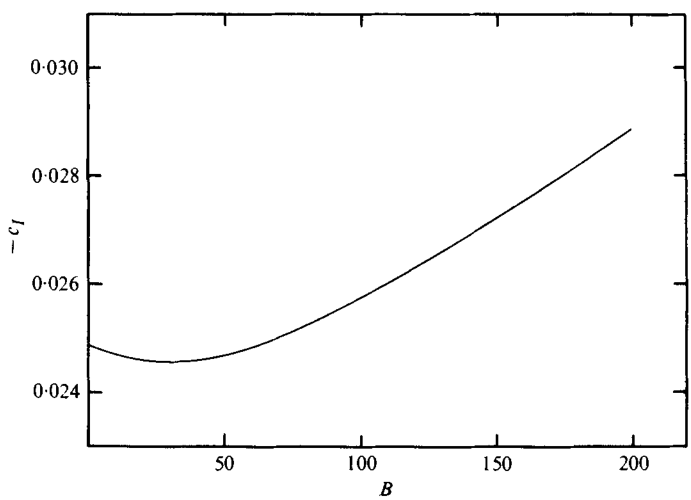

FIGURE 4. c_I as function of B for the next least stable mode. $\alpha = 1{\cdot}0$, $\beta = 0{\cdot}5$, $R = 2 \times 10^5$.

destabilization for small values of B but a strong stabilization for larger values of B. The destabilization can be regarded as insignificant and it seems likely that the fibres will stabilize higher modes as well.

6. Conclusions

The linear hydrodynamic stability problem for a plane Poiseuille flow of a fibre suspension, modelled according to Batchelor (1970*a*, *b*, 1971), has been solved. For a fixed, but not too large, wave angle, the critical Reynolds number increases monotonically with the product of the volume fraction of the fibres and the square of their aspect ratio. For large wave angles, the stabilizing effect of the fibres disappears for the mean profile considered.

The authors have benefited from some very fruitful discussions with Professor Mårten T. Landahl and Professor Louis N. Howard.

Appendix. The functions g_k

The functions g_k appearing in (3.2*c*–*d*) are given by

$$
\begin{aligned}
g_1(y) &= -DU, \\
g_2(y) &= a^{-1}[2(U-e)+r], \\
g_3(y) &= (U-c)\,a^{-1}[(U-e)(DU+bF)-rDU], \\
g_4(y) &= (U-c)^2a^{-2}[(U-e)\{(U-e)(U-e+r)+a[3bDF-k^2(U-c)+ak^4] \\
&\quad -(bqfd)\}-aDU\{DU+bF\}], \\
g_5(y) &= (U-c)^3a^{-2}[(bF)(U-e)^2-DU\{a[2bDF-k^2(U-c)-D^2U+ak^4]-(bqfd)\} \\
&\quad +(U-e)\{a(3bD^2F-2k^2DU-2D^3U)-bq(lDU+fF)\}], \\
g_6(y) &= (U-c)^4a^{-2}[(U-e)\{a(bD^3F-k^2D^2U-D^4U)-bq(lD^2U+fDF) \\
&\quad +(U-e)(bDF-k^2(U-c)-D^2U+ak^4)\} \\
&\quad -DU\{a(bD^2F-k^2DU-D^3U)-bq(lDU+fF)\}],
\end{aligned}
$$

where
$$a = \frac{i}{\alpha R}, \quad b = \frac{iB\alpha}{R}, \quad d = -\frac{\alpha^2}{k^2}, \quad f = -\frac{B\beta}{kR}, \quad l = -\frac{i\beta}{\alpha k},$$

$$q = \frac{i\alpha\beta}{k}, \quad r = \frac{i}{R}\left[-\frac{\alpha^3 B}{k^2}+\frac{\alpha\beta^2 B}{k^2}-\frac{k^2}{\alpha}\right], \quad F = \frac{DU}{U-c}.$$

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